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TOPICS IN COMPLEX ANALYSIS
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SHEET 13

Exercises for 'Topics in complex analysis'

(10/12/2025)

H 13.1 (The Cauchy-Riemann equations and $\mathbb{C}$-linearity on $\mathbb{C}^n$)

Identify $\mathbb{C}^n \simeq \mathbb{R}^{2n}$ and let $U \subset \mathbb{C}^n$. Consider a differentiable function $f : U \to \mathbb{C}$. By definition, at each point $a \in U$ there exists some $\mathbb{R}$-linear map $Df(a) : \mathbb{R}^{2n} \to \mathbb{C}$ such that

$$\lim_{\substack{h\to 0\\ h\neq 0}} \frac{|f(a+h) - f(a) - Df(a)h|}{\|h\|} = 0.$$

Show that $Df(a)$ is $\mathbb{C}$-linear if and only if for all $1 \leq j \leq n$ we have

$$\frac{\partial}{\partial \bar{z}_j} f(a) = 0,$$

where $\frac{\partial}{\partial \bar{z}_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\right)$ and $z = x + iy$ with $x, y \in \mathbb{R}^n$.

Remark: If $f : U \to \mathbb{C}$ is only assumed to be C^1 then it is differentiable (by standard real analysis), so the conclusion of the exercise holds. This proves the equivalence of Remark 9.2 (ii) and Definition 9.1. In fact one could even weaken the assumption that f is C^1 to locally integrable (and f satisfies the Cauchy-Riemann equations in the sense of distributions), since the decomposition $\Delta = 4\sum_{j=1}^n \frac{\partial}{\partial z_j}\frac{\partial}{\partial \overline{z_j}}$ of the Laplacian operator gives $\Delta f = 0$, which by elliptic regularity implies that f can be corrected on a set of measure zero to become analytic, and this exercise applies.

H 13.2 (The slicing method in action)

In this exercise we transfer some well-known results from single-variable complex analysis to the multivariable setting. Show the following.

a) Liouville's theorem: Every bounded entire function $f : \mathbb{C}^n \to \mathbb{C}$ is constant.

b) Identity theorem: Let $D \subset \mathbb{C}^n$ be a domain and $f : D \to \mathbb{C}$ be holomorphic. If $f|_{B_r(a)} \equiv 0$ for some $a \in D$ and $r > 0$, then $f \equiv 0$.

c) Open mapping theorem: Let $D \subset \mathbb{C}^n$ be a domain and $f : D \to \mathbb{C}$ be non-constant and holomorphic. Then $f(D)$ is again a domain.

d) Maximum principle: Let $D \subset \mathbb{C}^n$ be a domain and $f : D \to \mathbb{C}$ be holomorphic. If $|f|$ attains its maximum on D then f is constant.

H 13.3 (Failure of the open mapping theorem in the fully vectorial case)

In H 13.2 we proved the open mapping theorem for functions $f : D \to \mathbb{C}$. Here we show that it is false for vectorial functions $f : D \to \mathbb{C}^m$ for $m \geq 2$ even when no component is constant. Define $f : \mathbb{C}^2 \to \mathbb{C}^2$ by $f(z_1, z_2) = (z_1, z_1 z_2)$. Show that f is holomorphic but not an open map.
Hint: To guess where the map is not open, find where its differential is not invertible.

H 13.4 (On power series in several variables)

a) Determine for each series below the largest open set $U \subset \mathbb{C}^2$ where it converges absolutely. Is U convex?

(i) $\displaystyle\sum_{n=0}^{\infty} z^n w^n,$ (ii) $\displaystyle\sum_{n=1}^{\infty} z^n w^{n!}.$

b) Let $F(z) = \sum_{\alpha \in (\mathbb{N}_0)^n} c_\alpha z^\alpha$ be a formal power series centered at the origin. Show that if $z = (z_1, \ldots, z_n)$ is such that $F(z)$ converges absolutely, then $F(\lambda_1 z_1, \ldots, \lambda_n z_n)$ also converges absolutely if $|\lambda_i| \leq 1$ for all $1 \leq i \leq n$.